IHARA'S ZETA FUNCTION FOR PERIODIC GRAPHS AND ITS APPROXIMATION IN THE AMENABLE CASE

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Abstract. In this paper, we give a more direct proof of the results by Clair and Mokhtari-Sharghi [7] on the zeta functions of periodic graphs. In particular, using appropriate operator-algebraic techniques, we establish a determinant formula in this context and examine its consequences for the Ihara zeta function. Moreover, we answer in the affirmative one of the questions raised in [12] by Grigorchuk and Żuk. Accordingly, we show that the zeta function of a periodic graph with an amenable group action is the limit of the zeta functions of a suitable sequence of finite subgraphs.

0. Introduction

The zeta functions associated to finite graphs by Ihara [20], Hashimoto [15, 16], Bass [4] and others, combine features of Riemann's zeta function, Artin L-functions, and Selberg's zeta function, and may be viewed as analogues of the Dedekind zeta functions of a number field. They are defined by an Euler product and have an analytic continuation to a meromorphic function satisfying a functional equation. They can be expressed as the determinant of a perturbation of the graph Laplacian and, for Ramanujan graphs, satisfy a counterpart of the Riemann hypothesis [28]. Other relevant papers are [31, 17, 18, 27, 25, 10, 21, 29, 30, 19, 3, 22].

In differential geometry, researchers have first studied compact manifolds, then infinite covers of those, and finally, noncompact manifolds with greater complexity. Likewise, in the graph setting, one passes from finite graphs to infinite periodic graphs, and then possibly to other types of infinite graphs. In fact, the definition of the Ihara zeta function was extended to (countable) periodic graphs by Clair and Mokhtari-Sharghi [7], and a corresponding determinant formula was proved. They deduce this result as a specialization of the treatment of group actions on trees (the so-called theory of tree lattices, as developed by Bass, Lubotzky and others, see [5]). We mention [13] for a recent review of some results on zeta functions for finite or periodic simple graphs, and [12, 7, 8, 9] for the computation of the Ihara zeta function of several periodic simple graphs.

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In [12], Grigorchuk and Żuk defined zeta functions of infinite discrete groups, and of some class of infinite periodic graphs (which they call residually finite), and asked how to obtain the zeta function of a periodic graph by means of the zeta functions of approximating finite subgraphs, in the case of amenable or residually finite group actions.

The purpose of the present work is twofold: first, to give a different proof of the main result obtained by Clair and Mokhtari-Sharghi in [7]; second, to answer in the affirmative one of the questions raised by Grigorchuk and Żuk in [12].

As for the first point, some combinatorial results in Section 1 give a more direct proof of the determinant formula in Theorem 4.1. Moreover, the theory of analytic determinants developed in Section 3 allows us to use analytic functions instead of formal power series in that formula, as well as to establish functional equations for suitable completions of the Ihara zeta function, generalizing results contained in [13].

As for the second point, we take advantage of the technical framework developed in this paper to show, in the case of amenable group actions, that the Ihara zeta function is indeed the limit of the zeta functions of a suitable sequence of approximating finite graphs. For the sake of completeness, we mention that, in [8], Clair and Mokhtari-Sharghi have given a positive answer in the case of residually finite group actions.

This paper is organized as follows. We start in Section 1 by recalling some notions from graph theory and prove all the combinatorial results we need in the following sections. In Section 2, we then define the analogue of the Ihara zeta function and show that it is a holomorphic function in a suitable disc, while, in Section 4, we prove a corresponding determinant formula, which relates the zeta function with the Laplacian of the graph. The formulation and proof of this formula requires some care because it involves the definition and properties of a determinant for bounded operators (acting on an infinite dimensional Hilbert space and) belonging to a von Neumann algebra with a finite trace. This issue is addressed in Section 3. In Section 5, we establish several functional equations for various possible completions of the zeta function. In the final section, we prove the approximation result mentioned above.

In closing this introduction, we note that in [14] we define and study the Ihara zeta functions attached to a new class of infinite graphs, called self-similar fractal graphs, which have greater complexity than the periodic ones.

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1. Preliminary results

We recall some notions from graph theory, following [26]. A *graph* $X = (VX, EX)$ consists of a collection VX of objects, called *vertices*, and a collection EX of objects called (oriented) *edges*, together with two maps $e \in EX \mapsto (o(e), t(e)) \in VX \times VX$ and $e \in EX \mapsto \overline{e} \in EX$, satisfying the following conditions: $\overline{\overline{e}} = e$, $o(\overline{e}) = t(e)$, $\forall e \in EX$. The vertex $o(e)$ is called the *origin* of e, while $t(e)$ is called the *terminus* of e. The edge e is said to join the vertices $u := o(e)$, $v := t(e)$, while u and v are said to be *adjacent*, which is denoted $u \sim v$. The edge e is called a *loop* if $o(e) = t(e)$. The *degree* of a vertex v is $\deg(v) := |\{e \in EX : o(e) = v\}|$, where $|\cdot|$ denotes the cardinality. A *path* of length m in X from $u = o(e_1) \in VX$ to

$v = t(e_m) \in VX$ is a sequence of m edges $(e_1, \dots, e_m)$, where $o(e_{i+1}) = t(e_i)$, for $i = 1, ..., m-1$. In the following, the length of a path C is denoted by $|C|$. A path is *closed* if $u = v$. A graph is said to be *connected* if there is a path between any pair of distinct vertices.

The couple $\{e, \overline{e}\}$ is called a *geometric edge*. An *orientation* of X is the choice of one oriented edge for each couple, which is called *positively oriented*. Denote by E^+X the set of positively oriented edges. Then the other edge of each couple will be called negatively oriented, and denoted $\overline{e}$, if $e \in E^+X$. The set of negatively oriented edges is denoted E^-X. Then $EX = E^+X \cup E^-X$.

In this paper, we assume that the graph $X = (VX, EX)$ is connected, countable [*i.e.* VX and EX are countable sets] and with bounded degree [*i.e.* $d := \sup_{v \in VX} \deg(v) < \infty$]. We also choose, once and for all, an orientation of X.

Let Γ be a countable discrete subgroup of automorphisms of X, which acts

(1) without inversions, *i.e.* $\gamma(e) \neq \overline{e}, \forall \gamma \in \Gamma, e \in EX$,
(2) discretely, *i.e.* $\Gamma_v := \{\gamma \in \Gamma : \gamma v = v\}$ is finite, $\forall v \in VX$,
(3) with bounded covolume, *i.e.* $\mathrm{vol}(X/\Gamma) := \sum_{v \in \mathcal{F}_0} \frac{1}{|\Gamma_v|} < \infty$, where $\mathcal{F}_0 \subset VX$ contains exactly one representative for each equivalence class in VX/Γ.

We note that the above bounded covolume property is equivalent to

$$\mathrm{vol}(EX/\Gamma) := \sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|} < \infty,$$

where $\mathcal{F}_1 \subset EX$ contains exactly one representative for each equivalence class in EX/Γ.

Let us now define two useful unitary representations of Γ.

Denote by $\ell^2(VX)$ the Hilbert space of functions $f : VX \to \mathbb{C}$ such that $\|f\|^2 := \sum_{v \in VX} |f(v)|^2 < \infty$. A unitary representation of Γ on $\ell^2(VX)$ is given by $(\lambda_0(\gamma)f)(x) := f(\gamma^{-1}x)$, for $\gamma \in \Gamma$, $f \in \ell^2(VX)$, $x \in VX$. Then the von Neumann algebra $\mathcal{N}_0(X, \Gamma) := \{\lambda_0(\gamma) : \gamma \in \Gamma\}'$ of all the bounded operators on $\ell^2(VX)$ commuting with the action of Γ, inherits a trace given by

$$Tr_\Gamma(A) := \sum_{x \in \mathcal{F}_0} \frac{1}{|\Gamma_x|} A(x, x), \ A \in \mathcal{N}_0(X, \Gamma). \tag{1.1}$$

Analogously, denote by $\ell^2(EX)$ the Hilbert space of functions $\omega : EX \to \mathbb{C}$ such that $\|\omega\|^2 := \sum_{e \in EX} |\omega(e)|^2 < \infty$. A unitary representation of Γ on $\ell^2(EX)$ is given by $(\lambda_1(\gamma)\omega)(e) := \omega(\gamma^{-1}e)$, for $\gamma \in \Gamma$, $\omega \in \ell^2(EX)$, $e \in EX$. Then the von Neumann algebra $\mathcal{N}_1(X, \Gamma) := \{\lambda_1(\gamma) : \gamma \in \Gamma\}'$ of all the bounded operators on $\ell^2(EX)$ commuting with the action of Γ, inherits a trace given by

$$Tr_\Gamma(A) := \sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|} A(e, e), \ A \in \mathcal{N}_1(X, \Gamma). \tag{1.2}$$

At this stage, we need to introduce some additional terminology from graph theory.

Definition 1.1 (Reduced Paths)**.**
(i) A path $(e_1, \dots, e_m)$ has *backtracking* if $e_{i+1} = \overline{e}_i$, for some $i \in \{1, \dots, m-1\}$. A path with no backtracking is also called *proper*.

(ii) A closed path is called *primitive* if it is not obtained by going $n \geq 2$ times around some other closed path.
(iii) A proper closed path $C = (e_1, \dots, e_m)$ has a *tail* if there is $k \in \mathbb{N}$ such that $e_{m-j+1} = \overline{e}_j$, for $j = 1, \dots, k$. Denote by $\mathcal{C}$ the set of proper tail-less closed paths, also called *reduced* closed paths.

Definition 1.2 (Cycles)**.** Given closed paths $C = (e_1, \dots, e_m)$, $D = (e'_1, \dots, e'_m)$, we say that C and D are *equivalent*, and write $C \sim_o D$, if there is $k \in \mathbb{N}$ such that $e'_j = e_{j+k}$, for all j, where $e_{m+i} := e_i$, that is, the origin of D is shifted k steps with respect to the origin of C. The equivalence class of C is denoted $[C]_o$. An equivalence class is also called a *cycle*. Therefore, a closed path is just a cycle with a specified origin.

Denote by $\mathcal{R}$ the set of reduced cycles, and by $\mathcal{P} \subset \mathcal{R}$ the subset of primitive reduced cycles, also called *prime cycles*.

Definition 1.3 (Equivalence relation)**.**
(i) Given C, $D \in \mathcal{C}$, we say that C and D are Γ-*equivalent*, and write $C \sim_\Gamma D$, if there is an isomorphism $\gamma \in \Gamma$ such that $D = \gamma(C)$. We denote by $[\mathcal{C}]_\Gamma$ the set of Γ-equivalence classes of reduced closed paths.
(ii) Similarly, given C, $D \in \mathcal{R}$, we say that C and D are Γ-*equivalent*, and write $C \sim_\Gamma D$, if there is an isomorphism $\gamma \in \Gamma$ such that $D = \gamma(C)$. We denote by $[\mathcal{R}]_\Gamma$ the set of Γ-equivalence classes of reduced cycles, and analogously for the subset $\mathcal{P}$.

Remark 1.4. In the rest of the paper, we denote by $\mathcal{C}_m$ the subset of $\mathcal{C}$ consisting of closed paths of length m. An analogous meaning is attached to $\mathcal{R}_m$ and $\mathcal{P}_m$.

Our proof of formula *(iv)* in Theorem 2.2 requires a generalization of a result by Kotani and Sunada [21] to infinite covering graphs. This is done in Proposition 1.6, whose proof depends on a new combinatorial result contained in Lemma 1.5.

Define the *effective length* of a cycle C, denoted by $\ell(C)$, as the length of the prime cycle underlying C, and observe that $\ell(C)$ is constant on the Γ-equivalence class of C. Therefore, if $\xi \in [\mathcal{R}]_\Gamma$, we can define $\ell(\xi) := \ell(C)$, for any representative $C \in \xi$. Recall that, for any cycle C, the *stabilizer* of C in Γ is the subgroup $\Gamma_C := \{\gamma \in \Gamma : \gamma(C) = C\}$. Moreover, if C_1, $C_2 \in \xi$, then the stabilizers Γ_{C_1}, Γ_{C_2} are conjugate subgroups in Γ, and we denote by $\mathcal{S}(\xi)$ their common cardinality.

For the purposes of the next few results, for any closed path $D = (e_0, \dots, e_{m-1})$, we also denote e_j by $e_j(D)$.

Lemma 1.5. *Let* $\xi \in [\mathcal{R}_m]_\Gamma$. *Then*

$$\sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|} |\, \{D \in \mathcal{C}_m : [D]_{o,\Gamma} = \xi, e_0(D) = e\} \,| = \frac{\ell(\xi)}{\mathcal{S}(\xi)}.$$

Proof. Let us first observe that, if C_1, $C_2 \in \xi$, then $\cap_{e \in EC_1} \Gamma_e$ is conjugate in Γ to $\cap_{e \in EC_2} \Gamma_e$, and we denote by $\mathcal{I}(\xi)$ their common cardinality.

Let $C \in \mathcal{R}_m$ be such that $[C]_\Gamma = \xi$. By choosing each time a different starting edge, we obtain $\ell := \ell(C) \equiv \ell(\xi)$ closed paths from C. Denote them by $D_1, \dots, D_\ell$, and observe that any two of them can be Γ-equivalent, *i.e.* $D_i = \gamma(D_j)$, for some $\gamma \in \Gamma$, if and only if $\gamma \in \Gamma_C$. Moreover, if $\gamma \in \cap_{e \in EC} \Gamma_e \subset \Gamma_C$, then $\gamma(D_i) = D_i$, for $i = 1, \dots, \ell$. Therefore, there are only $k \equiv k(\xi) := \frac{\ell(\xi)\mathcal{I}(\xi)}{\mathcal{S}(\xi)}$ distinct Γ-classes of closed paths generated by the D_i's, and we denote them by $\pi_1, \dots, \pi_k$.

Let π be one of them, and observe that, for any $e \in \mathcal{F}_1$, there are either no closed paths D representing π and such that $e_0(D) = e$, or there are $\frac{|\Gamma_e|}{\mathcal{I}(\xi)}$ distinct closed paths D representing π and such that $e_0(D) = e$. Indeed, if there is a closed path D representing π and such that $e_0(D) = e$, then any $\gamma \in \Gamma_e$ generates a closed path $\gamma(D)$ representing π and such that $e_0(\gamma(D)) = e$, but, if $\gamma \in \cap_{e \in ED}\Gamma_e$, then $\gamma(D) = D$. Hence, the claim is established.

Let us now introduce a discrete measure on $\mathcal{F}_1$. Let us say that a Γ-class of closed paths π starts at $e \in \mathcal{F}_1$ if there is $D \in \pi$ such that $e_0(D) = e$. Let us set, for $e \in \mathcal{F}_1$, $\mu_\xi(e) = 1$, if e is visited by some π_i, $i = 1, \ldots, k$, and $\mu_\xi(e) = 0$, otherwise. It is easy to see that μ_ξ depends only on ξ and is in particular independent of the representative C. Observe that $\mu_\xi(\mathcal{F}_1) = k(\xi)$.

Therefore, for any $e \in \mathcal{F}_1$, we get

$$|\{D \in \mathcal{C}_m : [D]_{o,\Gamma} = \xi, e_0(D) = e\}| = \frac{\mu_\xi(e) \cdot |\Gamma_e|}{\mathcal{I}(\xi)},$$

and, finally,

$$\sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|} |\{D \in \mathcal{C}_m : [D]_{o,\Gamma} = \xi, e_0(D) = e\}| = \frac{1}{\mathcal{I}(\xi)} \sum_{e \in \mathcal{F}_1} \mu_\xi(e) = \frac{k(\xi)}{\mathcal{I}(\xi)} = \frac{\ell(\xi)}{\mathcal{S}(\xi)}.$$

□

Define, for $\omega \in \ell^2(EX)$, $e \in EX$,

$$(T\omega)(e) = \sum_{\substack{t(e') = o(e) \\ e' \neq \overline{e}}} \omega(e').$$

Then, we have

Proposition 1.6.
(i) $T \in \mathcal{N}_1(X, \Gamma)$, $\|T\| \leq d - 1$,
(ii) for $m \in \mathbb{N}$, $T^m e = \sum_{\substack{(e, e_1, \ldots, e_m) \\ \text{proper path}}} e_m$, *for* $e \in EX$,
(iii) $Tr_\Gamma(T^m) = N_m^\Gamma := \sum_{[C]_\Gamma \in [\mathcal{R}_m]_\Gamma} \frac{\ell([C]_\Gamma)}{\mathcal{S}([C]_\Gamma)}$, *the number of* Γ*-equivalence classes of reduced cycles of length* m. *Here,* Tr_Γ *is the trace on* $\mathcal{N}_1(X, \Gamma)$ *introduced in (1.2).*

Proof. (i), (ii) are easy to check.
(iii) Using Lemma 1.5, we obtain

$$\begin{aligned}
Tr_\Gamma(T^m) &= \sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|} T^m(\tilde{e}, \tilde{e}) \\
&= \sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|} \sum_{\substack{(e, e_1, \ldots, e_{m-1}, e) \\ \text{reduced path}}} 1 \\
&= \sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|} |\{C \in \mathcal{C}_m : e_0(C) = e\}| \\
&= \sum_{[C]_\Gamma \in [\mathcal{R}]_\Gamma} \sum_{e \in \mathcal{F}_1} \frac{1}{|\Gamma_e|} |\{D \in \mathcal{C}_m : [D]_0 \sim_\Gamma C, e_0(D) = e\}| \\
&= N_m^\Gamma.
\end{aligned}$$

□

2. The Zeta function

Before introducing the zeta function of an infinite periodic graph, we recall its definition for a finite $(q+1)$-regular graph X (*i.e.* such that $\deg(v) = q+1$, for all $v \in VX$). In that case, the Ihara zeta function Z_X is defined by an Euler product of the form

$$Z_X(u) := \prod_{C \in \mathcal{P}} (1 - u^{|C|})^{-1}, \quad \text{for } |u| < \frac{1}{q}, \tag{2.1}$$

where $\mathcal{P}$ is the set of prime cycles of X. By way of comparison, recall that the Riemann zeta function is given by the Euler product

$$\zeta(s) := \prod_p (1 - p^{-s})^{-1}, \quad \text{for } Re\ s > 1, \tag{2.2}$$

where p ranges over all the rational primes. To see the correspondence between Z_X and ζ, simply let $u := q^{-s}$ and observe that $u^{|C|} = (q^{|C|})^{-s}$. Also note that $|u| < \frac{1}{q}$ if and only if $Re\ s > 1$.

Let us now return to the case of periodic graphs and introduce the Ihara zeta function via its Euler product as well as show that this defines a holomorphic function in a suitable disc.

Definition 2.1 (Zeta function)**.** Let $Z(u) = Z_{X,\Gamma}(u)$ be given by

$$Z_{X,\Gamma}(u) := \prod_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} (1 - u^{|C|})^{-\frac{1}{|\Gamma_C|}},$$

for $u \in \mathbb{C}$ sufficiently small so that the infinite product converges.

In the following proposition we let

$$\det{}_\Gamma(B) := \exp \circ Tr_\Gamma \circ \log(B), \qquad \text{for } B \in \mathcal{N}_1(X, \Gamma).$$

We refer to Section 3 for more details. Formula (iv) in the following theorem was first established in [7], although with a different proof.

Theorem 2.2.
(i) $Z(u) := \prod_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} (1 - u^{|C|})^{-\frac{1}{|\Gamma_C|}}$ *defines a holomorphic function in the open disc* $\{u \in \mathbb{C} : |u| < \frac{1}{d-1}\}$.
(ii) $u \frac{Z'(u)}{Z(u)} = \sum_{m=1}^\infty N_m^\Gamma u^m$, *for* $|u| < \frac{1}{d-1}$.
(iii) $Z(u) = \exp\left(\sum_{m=1}^\infty \frac{N_m^\Gamma}{m} u^m\right)$, *for* $|u| < \frac{1}{d-1}$.
(iv) $Z(u) = \det_\Gamma(I - uT)^{-1}$, *for* $|u| < \frac{1}{d-1}$.

Proof. Observe that it follows from Proposition 1.6 that $\sum_{m=1}^\infty \frac{N_m^\Gamma}{m} u^m$ defines a function which is holomorphic in $\{u \in \mathbb{C} : |u| < \frac{1}{d-1}\}$. Moreover, for any $u \in \mathbb{C}$

such that $|u| < \frac{1}{d-1}$,

$$\begin{aligned}\sum_{m=1}^{\infty} N_m^{\Gamma} u^m &= \sum_{[C]_\Gamma \in [\mathcal{R}]_\Gamma} \frac{\ell([C]_\Gamma)}{\mathcal{S}([C]_\Gamma)} u^{|C|} \\ &= \sum_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} \sum_{m=1}^{\infty} \frac{|C|}{|\Gamma_C|} u^{|C^m|} \\ &= \sum_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} \frac{1}{|\Gamma_C|} \sum_{m=1}^{\infty} |C| u^{|C|m} \\ &= \sum_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} \frac{1}{|\Gamma_C|} u \frac{d}{du} \sum_{m=1}^{\infty} \frac{u^{|C|m}}{m} \\ &= - \sum_{[C]_\Gamma \in [\mathcal{P}]_\Gamma} \frac{1}{|\Gamma_C|} u \frac{d}{du} \log(1 - u^{|C|}) \\ &= u \frac{d}{du} \log Z(u),\end{aligned}$$

where, in the last equality, we have used uniform convergence on compact subsets of $\left\{ u \in \mathbb{C} : |u| < \frac{1}{d-1} \right\}$. From what has already been proved, $(i) - (iii)$ follow. Finally, for $|u| < \frac{1}{d-1}$, we have

$$\begin{aligned}\log Z(u) &= \sum_{m=1}^{\infty} \frac{N_m^{\Gamma}}{m} u^m \\ &= \sum_{m=1}^{\infty} \frac{1}{m} Tr_\Gamma((Tu)^m) \\ &= Tr_\Gamma \left(\sum_{m=1}^{\infty} \frac{(Tu)^m}{m} \right) \\ &= Tr_\Gamma(-\log(I - uT)).\end{aligned}$$

□

Example 2.3. Some examples of cycles with different stabilizers are shown in figures 2, 3. They refer to the graph in figure 1 which is the standard lattice graph $X = \mathbb{Z}^2$ endowed with the action of the group Γ generated by the reflection along the x-axis and the translations by elements $(m, n) \in \mathbb{Z}^2$, acting as $(m, n)(v_1, v_2) := (v_1 + 4m, v_2 + 4n)$, for $v = (v_1, v_2) \in VX = \mathbb{Z}^2$.

3. An analytic determinant for von Neumann algebras with a finite trace

In this section, we define a determinant for a suitable class of not necessarily normal operators in a von Neumann algebra with a finite trace. The results obtained are used in Section 4 to prove a determinant formula for the zeta function.

In a celebrated paper [11], Fuglede and Kadison defined a positive-valued determinant for finite factors (*i.e.* von Neumann algebras with trivial center and finite trace). Such a determinant is defined on all invertible elements and enjoys the main properties of a determinant function, but it is positive-valued. Indeed, for

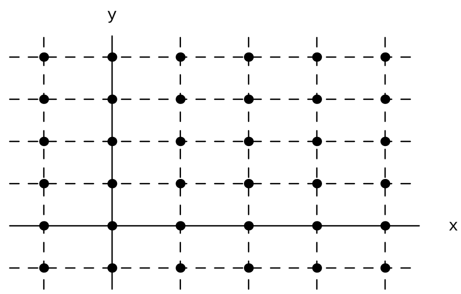

FIGURE 1. A periodic graph

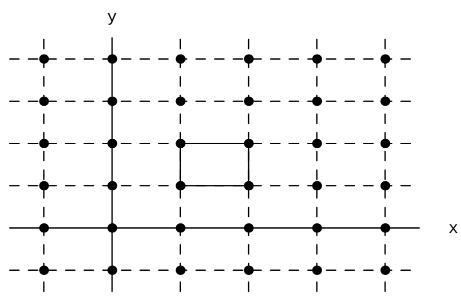

FIGURE 2. A cycle with $|\Gamma_C| = 1$

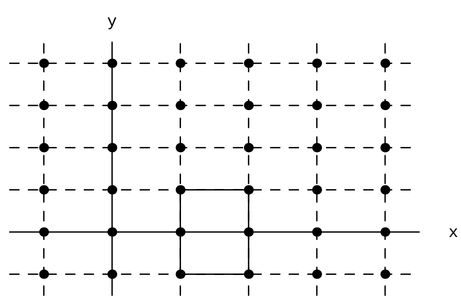

FIGURE 3. A cycle with $|\Gamma_C| = 2$

an invertible operator A with polar decomposition $A = UH$, where U is a unitary operator and $H := \sqrt{A^*A}$ is a positive self-adjoint operator, the Fuglede–Kadison determinant is defined by

$$Det(A) = \exp \circ \tau \circ \log H,$$

where $\log H$ may be defined via the functional calculus. Note, however, that the original definition was only given for a normalized trace.

For the purposes of the present paper, we need a determinant which is an analytic function. As we shall see, this can be achieved, but corresponds to a restriction of the domain of the determinant function and implies the loss of some important properties. In particular, the product formula of the Fuglede–Kadison determinant

only holds under certain restrictions in our case; see Propositions 3.4, 3.6, 3.7 and 3.8.

Let $(\mathcal{A},\tau)$ be a von Neumann algebra endowed with a finite trace. Then, a natural way to obtain an analytic function is to define, for $A\in\mathcal{A}$, $\det_\tau(A)=\exp\circ\tau\circ\log A$, where

$$\log(A) := \frac{1}{2\pi i}\int_\Gamma \log\lambda(\lambda-A)^{-1}d\lambda,$$

and Γ is the boundary of a connected, simply connected region Ω containing the spectrum of A. Clearly, once the branch of the logarithm is chosen, the integral above does not depend on Γ, provided Γ is given as above.

Then a naïve way of defining *det* is to allow all elements A for which there exists an Ω as above, and a branch of the logarithm whose domain contains Ω. Indeed, the following holds.

Lemma 3.1. *Let A, Ω, Γ be as above, and φ, ψ two branches of the logarithm such that both domains contain Ω. Then*

$$\exp\circ\tau\circ\varphi(A)=\exp\circ\tau\circ\psi(A).$$

Proof. The function $\varphi(\lambda)-\psi(\lambda)$ is continuous and everywhere defined on Γ. Since it takes its values in $2\pi i\mathbb{Z}$, it should be constant on Γ. Therefore,

$$\begin{aligned}\exp\circ\tau\circ\varphi(A)&=\exp\circ\tau\left(\frac{1}{2\pi i}\int_\Gamma 2\pi i n_0(\lambda-A)^{-1}d\lambda\right)\exp\circ\tau\circ\psi(A)\\&=\exp\circ\tau\circ\psi(A).\end{aligned}$$

□

The problem with the previous definition is its dependence on the choice of Ω. Indeed, it is easy to see that when $A=\begin{pmatrix}1&0\\0&i\end{pmatrix}$ and we choose Ω containing $\{e^{i\vartheta},\vartheta\in[0,\pi/2]\}$ and any suitable branch of the logarithm, we get $det(A)=e^{i\pi/4}$, if we use the normalized trace on 2×2 matrices. By contrast, if we choose Ω containing $\{e^{i\vartheta},\vartheta\in[\pi/2,2\pi]\}$ and a corresponding branch of the logarithm, we get $det(A)=e^{5i\pi/4}$. Therefore, we make the following choice.

Definition 3.2. Let $(\mathcal{A},\tau)$ be a von Neumann algebra endowed with a finite trace, and consider the subset $\mathcal{A}_0=\{A\in\mathcal{A}: 0\notin \operatorname{conv}\sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of A. For any $A\in\mathcal{A}_0$ we set

$$\det_\tau(A)=\exp\circ\tau\circ\left(\frac{1}{2\pi i}\int_\Gamma\log\lambda(\lambda-A)^{-1}d\lambda\right),$$

where Γ is the boundary of a connected, simply connected region Ω containing $\operatorname{conv}\sigma(A)$, and log is a branch of the logarithm whose domain contains Ω.

Corollary 3.3. *The determinant function defined above is well defined and analytic on $\mathcal{A}_0$.*

We collect several properties of our determinant in the following result.

Proposition 3.4. *Let $(\mathcal{A},\tau)$ be a von Neumann algebra endowed with a finite trace, and let $A\in\mathcal{A}_0$. Then*

(i) *$det_\tau(zA)=z^{\tau(I)}det_\tau(A)$, for any $z\in\mathbb{C}\setminus\{0\}$,*

(*ii*) *if A is normal, and $A = UH$ is its polar decomposition,*

$$det_\tau(A) = det_\tau(U) det_\tau(H),$$

(*iii*) *if A is positive, $det_\tau(A) = Det(A)$, where the latter is the Fuglede–Kadison determinant.*

Proof. (*i*) If, for a given $\vartheta_0 \in [0, 2\pi)$, the half-line $\{\rho e^{i\vartheta_0} \in \mathbb{C} : \rho > 0\}$ does not intersect $\operatorname{conv}\sigma(A)$, then the half-line $\{\rho e^{i(\vartheta_0+t)} \in \mathbb{C} : \rho > 0\}$ does not intersect $\operatorname{conv}\sigma(zA)$, where $z = re^{it}$. If log is the branch of the logarithm defined on the complement of the real negative half-line, then $\varphi(x) = i(\vartheta_0 - \pi) + \log(e^{-i(\vartheta_0-\pi)}x)$ is suitable for defining $\det_\tau(A)$, while $\psi(x) = i(\vartheta_0 + t - \pi) + \log(e^{-i(\vartheta_0+t-\pi)}x)$ is suitable for defining $\det_\tau(zA)$. Moreover, if Γ is the boundary of a connected, simply connected region Ω containing $\operatorname{conv}\sigma(A)$, then $z\Gamma$ is the boundary of a connected, simply connected region $z\Omega$ containing $\operatorname{conv}\sigma(zA)$. Therefore,

$$\begin{aligned}
\det_\tau(zA) &= \exp\circ\tau\left(\frac{1}{2\pi i}\int_{z\Gamma}\psi(\lambda)(\lambda - zA)^{-1}d\lambda\right)\\
&= \exp\circ\tau\left(\frac{1}{2\pi i}\int_{\Gamma}(i(\vartheta_0 + t - \pi) + \log(e^{-i(\vartheta_0+t-\pi)}re^{it}\mu))(\mu - A)^{-1}d\mu\right)\\
&= \exp\circ\tau\left((\log r + it)I + \frac{1}{2\pi i}\int_{\Gamma}\varphi(\mu)(\mu - A)^{-1}d\mu\right)\\
&= z^{\tau(I)}\det_\tau(A).
\end{aligned}$$

(*ii*) When $A = UH$ is normal, $U = \int_{[0,2\pi]} e^{i\vartheta}\ du(\vartheta)$, $H = \int_{[0,\infty)} r\ dh(r)$, then $A = \int_{[0,\infty)\times[0,2\pi]} re^{i\vartheta}\ d(h(r)\otimes u(\vartheta))$. The property $0 \notin \operatorname{conv}\sigma(A)$ is equivalent to the fact that the support of the measure $d(h(r)\otimes u(\vartheta))$ is compactly contained in some open half-plane

$$\{\rho e^{i\vartheta} : \rho > 0, \vartheta \in (\vartheta_0 - \pi/2, \vartheta_0 + \pi/2)\},$$

or, equivalently, that the support of the measure $dh(r)$ is compactly contained in $(0,\infty)$, and the support of the measure $du(\vartheta)$ is compactly contained in $(\vartheta_0 - \pi/2, \vartheta_0 + \pi/2)$. Therefore, $A \in \mathcal{A}_0$ is equivalent to $U, H \in \mathcal{A}_0$. Then

$$\log A = \int_{[0,\infty)\times(\vartheta_0-\pi/2,\vartheta_0+\pi/2)} (\log r + i\vartheta)\ d(h(r)\otimes u(\vartheta)),$$

which implies that

$$\begin{aligned}
\det_\tau(A) &= \exp\circ\tau\left(\int_0^\infty \log r\ dh(r) + \int_{\vartheta_0-\pi/2}^{\vartheta_0+\pi/2} i\vartheta\ du(\vartheta)\right)\\
&= \det_\tau(U)\cdot\det_\tau(H).
\end{aligned}$$

(*iii*) This follows by the argument given in (*ii*). □

Remark 3.5. We note that the above defined determinant function strongly violates the product property $\det_\tau(AB) = \det_\tau(A)\det_\tau(B)$. Indeed, the fact that $A, B \in \mathcal{A}_0$ does not imply $AB \in \mathcal{A}_0$, as is seen e.g. by taking $A = B = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Moreover, even if $A, B, AB \in \mathcal{A}_0$ and A and B commute, the product property may be violated, as is shown by choosing $A = B = \begin{pmatrix} 1 & 0 \\ 0 & e^{3i\pi/4} \end{pmatrix}$, and using the normalized trace on 2×2 matrices.

Proposition 3.6. *Let $(\mathcal{A}, \tau)$ be a von Neumann algebra endowed with a finite trace, and let $A, B \in \mathcal{A}$. Then, for sufficiently small $u \in \mathbb{C}$, we have*

$$det_\tau((I+uA)(I+uB)) = det_\tau(I+uA)det_\tau(I+uB).$$

Proof. The proof is inspired by that of Lemma 3 in [11]. Let us write $a := \log(I + uA)$, $b := \log(I+uB) \in \mathcal{A}$, and let $c(t) := e^{ta}e^b$, $t \in [0,1]$. As $\|a\| \le -\log(1 - |u|\|A\|)$, and $\|b\| \le -\log(1-|u|\|B\|)$, we get

$$\begin{aligned}
\|c(t) - 1\| &= \|e^{ta} - e^{-b}\|\|e^b\| \\
&\le e^{\|b\|}\left(e^{\|a\|} + e^{\|b\|} - 2\right) \\
&\le \frac{1}{1-|u|\|B\|}\left(\frac{1}{1-|u|\|A\|} + \frac{1}{1-|u|\|B\|} - 2\right) < 1,
\end{aligned}$$

for all $t \in [0,1]$, if we choose $|u|$ sufficiently small; hence, $c(t) \in \mathcal{A}_0$ for all $t \in [0,1]$. Now apply Lemma 2 in [11] which gives

$$\tau(\frac{d}{dt}\log c(t)) = \tau(c(t)^{-1}c'(t)) = \tau(e^{-b}e^{-ta}ae^{ta}e^b) = \tau(a).$$

Therefore, after integration for $t \in [0,1]$, we obtain $\tau(\log c(1)) - \tau(\log c(0)) = \tau(a)$, which means

$$\begin{aligned}
\tau\big(\log((I+uA)(I+uB))\big) &= \tau(\log c(1)) = \tau(a) + \tau(b) \\
&= \tau\big(\log(I+uA)\big) + \tau\big(\log(I+uB)\big),
\end{aligned}$$

and hence implies the claim. □

Proposition 3.7. *Let $(\mathcal{A}, \tau)$ be a von Neumann algebra endowed with a finite trace. Further, let $A \in \mathcal{A}$ have a bounded inverse, and let $T \in \mathcal{A}_0$. Then*

$$det_\tau(ATA^{-1}) = det_\tau T.$$

Proof. Indeed, for any polynomial p, we have $p(ATA^{-1}) = Ap(T)A^{-1}$. Applying the Stone–Weierstrass theorem on the compact set $\sigma(ATA^{-1}) = \sigma(T)$, we obtain $\log(ATA^{-1}) = A\log(T)A^{-1}$, from which the result follows. □

Proposition 3.8. *Let $(\mathcal{A}, \tau)$ be a von Neumann algebra endowed with a finite trace, and let $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \in Mat_2(\mathcal{A})$, with $T_{ii} \in \mathcal{A}$ such that $\sigma(T_{ii}) \subset B_1(1) := \{z \in \mathbb{C} : |z-1| < 1\}$, for $i = 1, 2$. Then*

$$det_\tau(T) = det_\tau(T_{11})det_\tau(T_{22}).$$

Proof. Indeed, for any $k \in \mathbb{N} \cup \{0\}$,

$$T^k = \begin{pmatrix} T_{11}^k & B_k \\ 0 & T_{22}^k \end{pmatrix},$$

for some $B_k \in \mathcal{A}$, so that, for any polynomial p,

$$p(T) = \begin{pmatrix} p(T_{11}) & B \\ 0 & p(T_{22}) \end{pmatrix},$$

for some $B \in \mathcal{A}$. It is easy to see that $\sigma(T) \subset \sigma(T_{11}) \cup \sigma(T_{22}) \subset B_1(1)$. Hence, applying the Stone–Weierstrass theorem on the compact set $\sigma(T)$, we obtain

$$\log(T) = \begin{pmatrix} \log(T_{11}) & C \\ 0 & \log(T_{22}) \end{pmatrix},$$

for some $C \in \mathcal{A}$. Therefore,

$\det_\tau(T) = \exp \circ\tau \circ \log(T) = \exp\big(\tau(\log(T_{11})) + \tau(\log(T_{22}))\big) = \det_\tau(T_{11})\det_\tau(T_{22})$,

as desired. $\square$

Corollary 3.9. *Let Γ be a discrete group, π_1, π_2 unitary representations of Γ, and τ_1, τ_2 finite traces on $\pi_1(\Gamma)'$ and $\pi_2(\Gamma)'$, respectively. Let $\pi := \pi_1 \oplus \pi_2$, $\tau := \tau_1 + \tau_2$, $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \in \pi(\Gamma)'$, with $\sigma(T_{ii}) \subset B_1(1) = \{z \in \mathbb{C} : |z-1| < 1\}$, for $i = 1, 2$. Then*

$$det_\tau(T) = det_{\tau_1}(T_{11}) det_{\tau_2}(T_{22}).$$

Proof. It is similar to the proof of Proposition 3.8. $\square$

4. The determinant formula

In this section, we prove the main result in the theory of the Ihara zeta functions, which says that Z is the reciprocal of a holomorphic function, which, up to a factor, is the determinant of a deformed Laplacian on the graph. We first need some technical results.

Let us denote by A the adjacency matrix of X, *i.e.* $(Af)(v) = \sum_{w\sim v} f(w)$, $f \in \ell^2(VX)$. Then (by [23], [24]) $\|A\| \leq d := \sup_{v\in VX} \deg(v) < \infty$, and it is easy to see that $A \in \mathcal{N}_0(X, \Gamma)$. Introduce $(Qf)(v) := (\deg(v) - 1)f(v)$, $v \in VX$, $f \in \ell^2(VX)$, and $\Delta(u) := I - uA + u^2Q \in \mathcal{N}_0(X, \Gamma)$, for $u \in \mathbb{C}$. Let us recall that $d := \sup_{v\in VX} \deg(v)$, and set $\alpha := \frac{d+\sqrt{d^2+4d}}{2}$. Then

Theorem 4.1 (Determinant formula)**.** *We have*

$$Z_{X,\Gamma}(u)^{-1} = (1-u^2)^{-\chi^{(2)}(X)} det_\Gamma(\Delta(u)), \quad for\ |u| < \frac{1}{\alpha},$$

where $\chi^{(2)}(X) := \sum_{v\in\mathcal{F}_0} \frac{1}{|\Gamma_v|} - \frac{1}{2}\sum_{e\in\mathcal{F}_1} \frac{1}{|\Gamma_e|}$ *is the L^2-Euler characteristic of (X, Γ), as introduced in* [6].

This theorem was first proved in [7] and is based on formula (iv) in Theorem 2.2 and the equality $\det_\Gamma(I - uT) = (1-u^2)^{-\chi^{(2)}(X)}\det_\Gamma(\Delta(u))$, for $|u| < \frac{1}{\alpha}$. The main difference with their proof is that we use an analytic determinant and operator-valued analytic functions instead of Bass' noncommutative determinant [4] and formal power series of operators.

We first prove two lemmas. Define, for $f \in \ell^2(VX)$, $\omega \in \ell^2(EX)$,

$$(\partial_0 f)(e) := f(o(e)),\ e \in EX$$
$$(\partial_1 f)(e) := f(t(e)),\ e \in EX$$
$$(\sigma\omega)(v) := \sum_{o(e)=v} \omega(e),\ v \in VX$$
$$(J\omega)(e) := \omega(\overline{e}),\ e \in EX,$$

and use the short-hand notation $I_V := Id_{\ell^2(VX)}$ and $I_E := Id_{\ell^2(EX)}$.

Lemma 4.2.
(i) $J\partial_1 = \partial_0$,
(ii) $\sigma\lambda_1(\gamma) = \lambda_0(\gamma)\sigma$, $\partial_i\lambda_0(\gamma) = \lambda_1(\gamma)\partial_i$, $i = 0, 1$, $\gamma \in \Gamma$,

(iii) $\sigma\partial_0 = I + Q$,
(iv) $\sigma\partial_1 = A$,
(v) $\partial_0\sigma = JT + I_E$,
(vi) $\partial_1\sigma = T + J$,
(vii) $(I_E - uJ)(I_E - uT) = (1-u^2)I_E - u\partial_1\sigma + u^2\partial_0\sigma$.

Proof. Let $f \in \ell^2(VX)$, $v \in VX$. Then

$$(\sigma\partial_0 f)(v) = \sum_{o(e)=v} (\partial_0 f)(e) = \sum_{o(e)=v} f(o(e)) = (1 + Q(v,v))f(v)$$
$$(\sigma\partial_1 f)(v) = \sum_{o(e)=v} (\partial_1 f)(e) = \sum_{o(e)=v} f(t(e)) = (Af)(v).$$

Moreover, for $\omega \in \ell^2(EX)$, $e \in EX$, we have

$$(\partial_1\sigma\omega)(e) = (\sigma\omega)(t(e)) = \sum_{o(e')=t(e)} \omega(e') = (T\omega)(e) + (J\omega)(e)$$
$$\partial_0\sigma = J\partial_1\sigma = J(T+J) = JT + I_E.$$

The rest of the proof is clear. □

Let us now consider the direct sum of the unitary representations λ_0 and λ_1, namely $\lambda(\gamma) := \lambda_0(\gamma) \oplus \lambda_1(\gamma) \in \mathcal{B}(\ell^2(VX) \oplus \ell^2(EX))$. Then, the von Neumann algebra $\lambda(\Gamma)' := \{S \in \mathcal{B}(\ell^2(VX) \oplus \ell^2(EX)) : S\lambda(\gamma) = \lambda(\gamma)S,\ \gamma \in \Gamma\}$ consists of operators $S = \begin{pmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{pmatrix}$, where $S_{ij}\lambda_j(\gamma) = \lambda_i(\gamma)S_{ij}$, $\gamma \in \Gamma$, $i,j = 0,1$, so that $S_{ii} \in \lambda_i(\Gamma)' \equiv \mathcal{N}_i(X,\Gamma)$, $i = 0,1$. Hence $\lambda(\Gamma)'$ inherits a trace given by

$$Tr_\Gamma \begin{pmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{pmatrix} := Tr_\Gamma(S_{00}) + Tr_\Gamma(S_{11}). \tag{4.1}$$

Introduce

$$\mathcal{L}(u) := \begin{pmatrix} (1-u^2)I_V & 0 \\ u\partial_0 - \partial_1 & I_E \end{pmatrix} \text{ and } \mathcal{M}(u) := \begin{pmatrix} I_V & u\sigma \\ u\partial_0 - \partial_1 & (1-u^2)I_E \end{pmatrix},$$

which both belong to $\lambda(\Gamma)'$. Then, we have

Lemma 4.3.
(i) $\mathcal{M}(u)\mathcal{L}(u) = \begin{pmatrix} \Delta(u) & u\sigma \\ 0 & (1-u^2)I_E \end{pmatrix}$,
(ii) $\mathcal{L}(u)\mathcal{M}(u) = \begin{pmatrix} (1-u^2)I_V & (1-u^2)u\sigma \\ 0 & (I_E - uJ)(I_E - uT) \end{pmatrix}$.
Moreover, for $|u|$ sufficiently small,
(iii) $\mathcal{L}(u)$, $\mathcal{M}(u)$ *are invertible, with a bounded inverse,*
(iv) $det_\Gamma(\mathcal{M}(u)\mathcal{L}(u)) = (1-u^2)^{Tr_\Gamma(I_E)} det_\Gamma(\Delta(u))$,
(v) $det_\Gamma(\mathcal{L}(u)\mathcal{M}(u)) = (1-u^2)^{Tr_\Gamma(I_V) - \frac{1}{2}Tr_\Gamma(I_E)} det_\Gamma(I_E - uT)$.

Proof. The formulas for $\mathcal{M}(u)\mathcal{L}(u)$ and $\mathcal{L}(u)\mathcal{M}(u)$ follow from the previous lemma. Moreover, for $|u|$ sufficiently small, $\sigma(\Delta(u))$, $\sigma((1-u^2)I_E)$, $\sigma((1-u^2)I_V)$ and $\sigma((I_E - uJ)(I_E - uT)) \subset B_1(1) = \{z \in \mathbb{C} : |z-1| < 1\}$, hence $\sigma(\mathcal{M}(u)\mathcal{L}(u))$ and $\sigma(\mathcal{L}(u)\mathcal{M}(u)) \subset B_1(1)$, as in the proof of Proposition 3.8. Therefore, $\mathcal{L}(u)$ and $\mathcal{M}(u)$

are invertible, with a bounded inverse, for $|u|$ sufficiently small. By Propositions 3.4 (i), 3.6 and Corollary 3.9, we obtain

$$\begin{aligned}\det_\Gamma(\mathcal{M}(u)\mathcal{L}(u)) &= \det_\Gamma(\Delta(u))\det_\Gamma((1-u^2)I_E)\\ &= (1-u^2)^{Tr_\Gamma(I_E)}\det_\Gamma(\Delta(u))\end{aligned}$$

and

$$\begin{aligned}\det_\Gamma(\mathcal{L}(u)\mathcal{M}(u)) &= \det_\Gamma((1-u^2)I_V)\det_\Gamma(I_E-uJ)\det_\Gamma(I_E-uT)\\ &= (1-u^2)^{Tr_\Gamma(I_V)}\det_\Gamma(I_E-uJ)\det_\Gamma(I_E-uT).\end{aligned}$$

Moreover, we have $\det_\Gamma(I_E-uJ) = (1-u^2)^{\frac{1}{2}Tr_\Gamma(I_E)}$. Indeed, using J to identify $\ell^2(E^-X)$ with $\ell^2(E^+X)$, we obtain a representation ρ of $\mathcal{B}(\ell^2(EX))$ onto $Mat_2(\mathcal{B}(\ell^2(E^+X)))$, under which $\rho(J) = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \rho(I_E) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Hence, by Propositions 3.6 and 3.8,

$$\begin{aligned}\det_\Gamma(I_E-uJ) &= \det_\Gamma(\rho(I_E-uJ))\\ &= \det_\Gamma\begin{pmatrix} I & -uI \\ -uI & I \end{pmatrix}\\ &= \det_\Gamma\left(\begin{pmatrix} I & uI \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -uI \\ -uI & I \end{pmatrix}\right)\\ &= \det_\Gamma\begin{pmatrix} (1-u^2)I & 0 \\ -uI & I \end{pmatrix}\\ &= (1-u^2)^{Tr_\Gamma(I)}\\ &= (1-u^2)^{\frac{1}{2}Tr_\Gamma(I_E)}.\end{aligned}$$

□

Proof (of Theorem 4.1).

Let us observe that, for sufficiently small $|u|$, we have

$$\mathcal{M}(u)\mathcal{L}(u) = \mathcal{M}(u)\mathcal{L}(u)\mathcal{M}(u)\mathcal{M}(u)^{-1},$$

so that, by Proposition 3.7, we get $\det_\Gamma(\mathcal{L}(u)\mathcal{M}(u)) = \det_\Gamma(\mathcal{M}(u)\mathcal{L}(u))$. Therefore, the claim follows from Lemma 4.3 (iv) and (v), equations (1.1) and (1.2) and Theorem 2.2. □

5. Functional equations

In this section, we obtain several functional equations for the Ihara zeta functions of $(q+1)$-regular graphs, *i.e.* graphs with $\deg(v) = q+1$, for any $v \in VX$, on which Γ acts freely [*i.e.* Γ_v is trivial, for $v \in VX$] and with finite quotient [*i.e.* $B := X/\Gamma$ is a finite graph]. The various functional equations correspond to different ways of completing the zeta functions, as is done in [28] for finite graphs. We extend here to non necessarily simple graphs the results contained in [13].

Lemma 5.1. *Let X be a $(q+1)$-regular graph, on which Γ acts freely and with finite quotient $B := X/\Gamma$. Let $\Delta(u) := (1+qu^2)I - uA$. Then*
(i) $\chi^{(2)}(X) = \chi(B) = |V(B)|(1-q)/2 \in \mathbb{Z}$,
(ii) $Z_{X,\Gamma}(u) = (1-u^2)^{\chi(B)}\, det_\Gamma((1+qu^2)I - uA)^{-1}$, *for* $|u| < \frac{1}{q}$,

(iii) by using the determinant formula in (ii), $Z_{X,\Gamma}$ can be extended to a function holomorphic at least in the open set

$$\Omega := \mathbb{R}^2 \setminus \left(\left\{ (x,y) \in \mathbb{R}^2 : x^2+y^2 = \frac{1}{q} \right\} \cup \left\{ (x,0) \in \mathbb{R}^2 : \frac{1}{q} \leq |x| \leq 1 \right\} \right).$$

See figure 4.

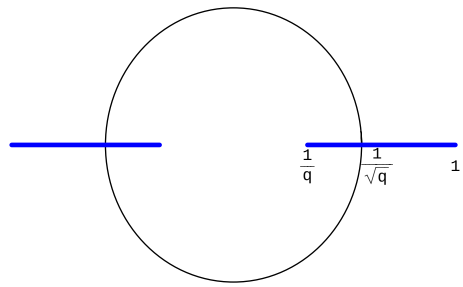

FIGURE 4. The open set Ω

(iv) $det_\Gamma\Big(\Delta(\frac{1}{qu})\Big) = (qu^2)^{-|VB|} det_\Gamma(\Delta(u))$, *for* $u \in \Omega \setminus \{0\}$.

Proof. (i) This follows by a simple computation.

(ii) This follows from (i).

(iii) Let us observe that

$$\sigma(\Delta(u)) = \left\{1+qu^2-u\lambda : \lambda \in \sigma(A)\right\} \subset \left\{1+qu^2-u\lambda : \lambda \in [-d,d]\right\}.$$

It follows that $0 \notin \operatorname{conv}\sigma(\Delta(u))$ at least for $u \in \mathbb{C}$ such that $1+qu^2-u\lambda \neq 0$ for $\lambda \in [-d,d]$, that is for $u=0$ or $\frac{1+qu^2}{u} \notin [-d,d]$, or equivalently, at least for $u \in \Omega$. The rest of the proof follows from Corollary 3.3.

(iv) This follows from Proposition 3.4 (i) and the fact that $Tr_\Gamma(I_V) = |VB|$. $\square$

The question whether the extension of the domain of $Z_{X,\Gamma}$ by means of the determinant formula is compatible with an analytic extension from the defining domain is a non-trivial issue, see the recent paper by Clair [9].

Theorem 5.2 (Functional equations)**.** *Let X be a $(q+1)$-regular graph, on which Γ acts freely and with finite quotient $B := X/\Gamma$. Then, for all $u \in \Omega$, we have*

(i) $\Lambda_{X,\Gamma}(u) := (1-u^2)^{-\chi(B)}(1-u^2)^{|VB|/2}(1-q^2u^2)^{|VB|/2}Z_{X,\Gamma}(u) = -\Lambda_{X,\Gamma}\Big(\frac{1}{qu}\Big)$,

(ii) $\xi_{X,\Gamma}(u) := (1-u^2)^{-\chi(B)}(1-u)^{|VB|}(1-qu)^{|VB|}Z_{X,\Gamma}(u) = \xi_{X,\Gamma}\Big(\frac{1}{qu}\Big)$,

(iii) $\Xi_{X,\Gamma}(u) := (1-u^2)^{-\chi(B)}(1+qu^2)^{|VB|}Z_{X,\Gamma}(u) = \Xi_{X,\Gamma}\Big(\frac{1}{qu}\Big)$.

Proof. They all follow from Lemma 5.1 (iv) by a straightforward computation. We prove (i) as an example.

$$\begin{aligned}\Lambda_X(u) &= (1-u^2)^{|VB|/2}(1-q^2u^2)^{|VB|/2}\det_\Gamma(\Delta(u))^{-1} \\ &= u^{|VB|}\Big(\frac{q^2}{q^2u^2}-1\Big)^{|VB|/2}(qu)^{|VB|}\Big(\frac{1}{q^2u^2}-1\Big)^{|VB|/2}\frac{1}{(qu^2)^{|VB|}}\det_\Gamma\Big(\Delta(\frac{1}{qu})\Big)^{-1} \\ &= -\Lambda_X\Big(\frac{1}{qu}\Big).\end{aligned}$$

□

Remark 5.3. Recall that a key property of the Riemann zeta function ζ is that its meromorphic continuation satisfies a functional equation $\xi(s) = \xi(1-s)$, for all $s \in \mathbb{C}$, where $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ denotes the completion of ζ and Γ is the usual Gamma function. Likewise, in Theorem 5.2, any of the functional equations relates the values of the corresponding completed Ihara zeta function at s and $1-s$, provided we set $u = q^{-s}$, as was explained at the beginning of Section 2. Note that $\frac{1}{qu} = \frac{1}{q^{1-s}}$.

6. Approximation by finite graphs in the amenable case

In this section, we show that the zeta function of a graph, endowed with a free and cofinite action of a discrete amenable group of automorphisms, is the limit of the zeta functions of a (suitable) sequence of finite subgraphs, thus answering in the affirmative a question raised by Grigorchuk and Żuk in [12].

Before doing that, we establish a result which is considered folklore by specialists. Roughly speaking, it states that a Γ-space is amenable if Γ is an amenable group, where a space is said to be amenable if it possesses a regular exhaustion. Such a result was stated by Cheeger and Gromov in [6] for CW-complexes and was proved by Adachi and Sunada in [2] for covering manifolds. We give here a proof in the case of covering graphs.

Throughout this section, X is a connected, countably infinite graph, and Γ is a countable discrete amenable group of automorphisms of X, which acts on X freely [i.e., any $\gamma \neq id$ has no fixed-points], and cofinitely [i.e., $B := X/\Gamma$ is a finite graph].

A fundamental domain for the action of Γ on X can be constructed as follows. Let $B = (VB, EB)$ be the quotient graph, and $p : X \to B$ the covering map. Let $EB = \{e_1, \ldots, e_k\}$, where the edges have been ordered in such a way that, for each $i \in \{1, \ldots, k\}$, e_i has at least a vertex in common with some e_j, with $j < i$. Choose $\tilde{e}_1 \in EX$ such that $p(\tilde{e}_1) = e_1$. Assume $\tilde{e}_1, \ldots, \tilde{e}_i$ have already been chosen in such a way that $p(\tilde{e}_j) = e_j$, for $j = 1, \ldots, i$, and, for any such j, $\tilde{e}_j$ has at least a vertex in common with some e_h, with $h < j$. Let $e_{i+1} \in EB$ have a vertex in common with e_j, for some $j \in \{1, \ldots, i\}$ and choose $\tilde{e}_{i+1} \in VX$ such that $p(\tilde{e}_{i+1}) = e_{i+1}$ and $\tilde{e}_{i+1}$ has a vertex in common with $\tilde{e}_j$. This completes the induction. Let $EF := \{\tilde{e}_1, \ldots, \tilde{e}_k\}$ and $VF := \{o(\tilde{e}_1), \ldots, o(\tilde{e}_k)\} \cup \{t(\tilde{e}_1), \ldots, t(\tilde{e}_k)\}$, so that $F = (VF, EF)$ is a connected finite subgraph of X which does not contain any Γ-equivalent edges. Then, F is said to be a *fundamental domain* for the action of Γ on X.

Definition 6.1. Let X be a countably infinite graph and Γ a countable discrete amenable group of automorphisms of X, which acts on X freely and cofinitely; further, let F be a corresponding fundamental domain. A sequence $\{K_n : n \in \mathbb{N}\}$ of finite subgraphs of X is called an *amenable exhaustion* of X if the following conditions hold:
(i) $K_n = \cup_{\gamma \in E_n} \gamma F$, where $E_n \subset \Gamma$, for all $n \in \mathbb{N}$,
(ii) $\cup_{n \in \mathbb{N}} K_n = X$,
(iii) $K_n \subset K_{n+1}$, for all $n \in \mathbb{N}$,
(iv) if $\mathcal{F}K_n := \{v \in VK_n : d(v, VX \setminus VK_n) = 1\}$, then $\displaystyle\lim_{n\to\infty} \frac{|\mathcal{F}K_n|}{|VK_n|} = 0$.

Then X is called an *amenable graph* if it possesses an amenable exhaustion.

Theorem 6.2. *Let X be a connected, countably infinite graph, Γ be a countable discrete amenable subgroup of automorphisms of X which acts on X freely and cofinitely and let F be a corresponding fundamental domain. Then X is an amenable graph.*

Proof. The proof is an adaptation of a proof by Adachi and Sunada in the manifold case, see [2].

The finite set $A := \{\gamma \in \Gamma : dist(\gamma F, F) \leq 1\}$ is symmetric [*i.e.* $\gamma \in A \iff \gamma^{-1} \in A$], generates Γ as a group, and contains the unit element. Introduce the Cayley graph $\mathcal{C}(\Gamma, A)$, whose vertices are the elements of Γ, and, by definition, there is one edge from γ_1 to γ_2 iff $\gamma_1^{-1}\gamma_2 \in A$. A subset $E \subset V\mathcal{C}(\Gamma, A)$ is said to be connected if, for any pair of distint vertices of E, there is a path in $\mathcal{C}(\Gamma, A)$, joining those two vertices, and consisting only of vertices of E.

From [1], Theorem 4, it follows that there is a sequence $\{E_j\}_{j\in\mathbb{N}}$ of connected finite subsets of Γ such that

$$\cup_{j\in\mathbb{N}} E_j = \Gamma, \qquad E_j \subset E_{j+1},\ \forall j \in \mathbb{N},$$
$$\frac{|E_j \cdot A \setminus E_j|}{|E_j|} \leq \frac{1}{j|A|},\ \forall j \in \mathbb{N},$$

where, for any $U_1, U_2 \subset \Gamma$, we set $U_1 \cdot U_2 = \{\gamma_1\gamma_2 : \gamma_i \in U_i, i = 1, 2\}$.

For each $n \in \mathbb{N}$, let $K_n := \cup_{\gamma\in E_n} \gamma F$. Then $\{K_n : n \in \mathbb{N}\}$ satisfies the claim. Indeed, let $b := |VF|$ and $a := |\mathcal{F}_0|$, so that $a|E_n| \leq |VK_n| \leq b|E_n|$, $n \in \mathbb{N}$. Moreover, for any $n \in \mathbb{N}$, we have

$$\mathcal{F}K_n \subset \cup_{\gamma\in U_n} \gamma F,$$

where $U_n := \{\gamma \in E_n :$ there is $\delta \in A$ such that $\gamma\delta \notin E_n\}$. Indeed, let $v \in \mathcal{F}K_n$ and $w \in VX \setminus VK_n$ be such that $d(v, w) = 1$. Then, there are $\gamma_0, \gamma_1 \in \Gamma$, $v_0, v_1 \in VF$, such that $v = \gamma_0 v_0$ and $w = \gamma_1 v_1$. Moreover, we have $\gamma_0 \in E_n$ and $\gamma_1 \notin E_n$. Let $\delta := \gamma_0^{-1}\gamma_1$, so that $dist(F, \delta F) = dist(\gamma_0 F, \gamma_1 F) \leq d(v, w) = 1$, which implies that $\delta \in A$. Hence, $\gamma_0 \in U_n$, and the claim follows.

Finally,

$$\begin{aligned}
|\mathcal{F}K_n| &\leq |U_n| \cdot |F| \\
&\leq b \sum_{\delta\in A} |E_n \setminus E_n \cdot \delta^{-1}| \\
&= b \sum_{\delta\in A} |E_n \cdot \delta \setminus E_n| \\
&\leq b|A| \cdot |E_n \cdot A \setminus E_n| \\
&\leq \frac{b}{n}|E_n| \leq \frac{b}{an}|VK_n|,
\end{aligned}$$

so condition (*iii*) of Definition 6.1 is satisfied, showing that $\{K_n : n \in \mathbb{N}\}$ is an amenable exhaustion. Hence, X is amenable, as desired. □

If $\Omega \subset VX$, $r \in \mathbb{N}$, we write $B_r(\Omega) := \{v' \in VX : \rho(v', v) \leq r\}$, where ρ is the geodesic metric on VX.

Lemma 6.3. *Let (X, Γ, F) be as above. Let $d := \sup_{v\in VX} \deg(v) < \infty$. Let $\{K_n\}$ be an amenable exhaustion of X, and $\varepsilon_n := \frac{|\mathcal{F}K_n|}{|VK_n|} \to 0$. Then, for any $r \in \mathbb{N}$, $|B_r(\mathcal{F}K_n)| \leq (d+1)^r \varepsilon_n |VK_n|$.*

Proof. Since

$$B_{r+1}(v) = \bigcup_{v' \in B_r(v)} B_1(v'),$$

we have $|B_{r+1}(v)| \leq (d+1)|B_r(v)|$, giving $|B_r(v)| \leq (d+1)^r$, $\forall v \in VX$, $r \geq 0$. As a consequence, for any finite set $\Omega \subset VX$, we have $B_r(\Omega) = \cup_{v' \in \Omega} B_r(v')$, giving

$$|B_r(\Omega)| \leq |\Omega|(d+1)^r, \quad \forall r \geq 0. \tag{6.1}$$

Therefore, $|B_r(\mathcal{F}K_n)| \leq (d+1)^r |\mathcal{F}K_n| = (d+1)^r \varepsilon_n |VK_n|$. □

Lemma 6.4. *Let (X, Γ, F) be as above. Let $\{K_n\}$ be an amenable exhaustion of X. Then, for any $B \in \mathcal{N}_0(X, \Gamma)$, we have*

$$\lim_{n \to \infty} \frac{Tr(P(K_n)BP(K_n))}{|VK_n|} = \frac{1}{|\mathcal{F}_0|}\ Tr_\Gamma(B),$$

where $P(K_n)$ is the orthogonal projection of $\ell^2(VX)$ onto $\ell^2(VK_n)$.

Proof. Denote by $\mathcal{F}_0$ a subset of VF consisting of one representative vertex for each Γ-class, and let $\mathcal{F}' := VF \setminus \mathcal{F}_0$ and $\delta := diam F$. Then, for any $n \in \mathbb{N}$, $VK_n = \sqcup_{\gamma \in E_n} \gamma \mathcal{F}_0 \sqcup \Omega_n$, where $\sqcup$ denotes "disjoint union" and $\Omega_n \subset B_\delta(\mathcal{F}K_n)$. Indeed, if $v \in \Omega_n := VK_n \setminus \sqcup_{\gamma \in E_n} \gamma \mathcal{F}_0$, then there is a unique $\gamma \in \Gamma$ such that $v \in \gamma \mathcal{F}_0$, so that $\gamma \notin E_n$, which implies $\gamma F \cap (VX \setminus VK_n) \neq \emptyset$, and $d(v, VX \setminus VK_n) \leq \delta$, which is the claim. Therefore,

$$\begin{aligned} Tr(P(K_n)B) &= \sum_{v \in VK_n} B(v,v) \\ &= \sum_{\gamma \in E_n} \sum_{v \in \mathcal{F}_0} B(\gamma v, \gamma v) + \sum_{v \in \Omega_n} B(v,v) \\ &= \sum_{\gamma \in E_n} \sum_{v \in \mathcal{F}_0} B(v,v) + \sum_{v \in \Omega_n} B(v,v) \\ &= |E_n| Tr_\Gamma(B) + \sum_{v \in \Omega_n} B(v,v). \end{aligned}$$

Moreover,

$$\left| \sum_{v \in \Omega_n} B(v,v) \right| \leq \|B\| |\Omega_n| \leq \|B\| |B_\delta(\mathcal{F}K_n)| \leq (d+1)^\delta \|B\| \varepsilon_n |VK_n|,$$

so that

$$\lim_{n \to \infty} \frac{\sum_{v \in \Omega_n} B(v,v)}{|VK_n|} = 0.$$

Besides,

$$\lim_{n \to \infty} \frac{|E_n|}{|VK_n|} = \frac{1}{|\mathcal{F}_0|},$$

because $|VK_n| = |E_n| \cdot |\mathcal{F}_0| + |\Omega_n|$. The claim follows. □

Lemma 6.5. *Let (X, Γ) be as above. Let A and Q be as in Section 4. Let $f(u) := Au - Qu^2$, for $u \in \mathbb{C}$. Then $\|f(u)\| < \frac{1}{2}$, for $|u| < \frac{1}{d + \sqrt{d^2 + 2(d-1)}}$.*

Proof. This follows from the estimate

$$\|f(u)\| \leq |u| \|A\| + |u|^2 \|Q\| \leq d|u| + (d-1)|u|^2,$$

which is valid for any $u \in \mathbb{C}$. □

Theorem 6.6 (Approximation by finite graphs). *Let X be a connected, countably infinite graph, and let Γ be a countable discrete amenable subgroup of automorphisms of X, which acts on X freely and cofinitely, and let F be a corresponding fundamental domain. Let $\{K_n : n \in \mathbb{N}\}$ be an amenable exhaustion of X. Then*

$$Z_{X,\Gamma}(u) = \lim_{n\to\infty} Z_{K_n}(u)^{\frac{|\mathcal{F}_0|}{|K_n|}},$$

uniformly on compact subsets of $\left\{u \in \mathbb{C} : |u| < \frac{1}{d+\sqrt{d^2+2(d-1)}}\right\}$.

Proof. For a finite subset $N \subset VX$, denote by $P(N) \in \mathcal{B}(\ell^2(VX))$ the orthogonal projection of $\ell^2(VX)$ onto span(N). Observe that, since N is an orthonormal basis for $\ell^2(N)$, we have $Tr\big(P(N)\big) = |N|$.

Let $f(u) := Au - Qu^2$ and $P_n := P(VK_n)$. Then

$$\log Z_{K_n}(u) = -\frac{1}{2} Tr(P_n(Q-I)P_n)\log(1-u^2) - Tr\log(P_n(I-f(u))P_n).$$

Moreover,

$$Tr\log(P_n(I-f(u))P_n) = -\sum_{k=1}^{\infty}\frac{1}{k} Tr\big((P_n f(u) P_n)^k\big).$$

Observe that, for $k \geq 2$,

$$\begin{aligned} Tr\big(P_n f(u)^k P_n\big) &= Tr\big(P_n(f(u)(P_n + P_n^{\perp}))^k P_n\big) \\ &= Tr\big((P_n f(u) P_n)^k\big) + \sum_{\substack{\sigma\in\{-1,1\}^{k-1} \\ \sigma\neq\{1,1,\dots,1\}}} Tr\big(P_n \prod_{j=1}^{k-1}[f(u)P_n^{\sigma_j}]f(u)P_n\big), \end{aligned}$$

where P_n^{-1} stands for $P_n^{\perp}$, the projection onto the orthogonal complement of $\ell^2(VK_n)$ in $\ell^2(VX)$, and

$$\begin{aligned} |Tr\big(P_n \prod_{j=1}^{k-1}[f(u)P_n^{\sigma_j}]f(u)P_n\big)| &= |Tr\big(\dots P_n f(u) P_n^{\perp}\dots\big)| \\ &\leq \|f(u)\|^{k-1} Tr(|P_n f(u) P_n^{\perp}|). \end{aligned}$$

Moreover, with $\Omega_n := B_1(VK_n) \setminus VK_n \subset B_1(\mathcal{F}K_n)$, we have

$$\begin{aligned} Tr(|P_n f(u) P_n^{\perp}|) &= Tr(|P(K_n) f(u) P(\Omega_n)|) \\ &\leq \|f(u)\| Tr(P(\Omega_n)) \\ &= \|f(u)\| |\Omega_n| \\ &\leq \|f(u)\|(d+1)\varepsilon_n |VK_n|. \end{aligned}$$

Therefore, we obtain

$$|Tr\big(P_n f(u)^k P_n\big) - Tr\big((P_n f(u) P_n)^k\big)| \leq (2^{k-1}-1)\|f(u)\|^k (d+1)\varepsilon_n |VK_n|,$$

so that

$$\begin{aligned}\Big|Tr\log(P_n(I-f(u))P_n)-Tr(P_n\log(I-f(u))P_n)\Big|\\ &=\Big|\sum_{k=1}^{\infty}\frac{1}{k}Tr\big((P_nf(u)P_n)^k\big)-\sum_{k=1}^{\infty}\frac{1}{k}Tr\big(P_nf(u)^kP_n\big)\Big|\\ &\leq\Big(\sum_{k=1}^{\infty}\frac{2^{k-1}\|f(u)\|^k}{k}\Big)(d+1)\varepsilon_n|VK_n|\\ &\leq C(d+1)\varepsilon_n|VK_n|,\end{aligned}$$

where the series converges for $|u|<\frac{1}{d+\sqrt{d^2+2(d-1)}}$, by Lemma 6.5. Hence,

$$\left|\frac{Tr\log(P_n(I-f(u))P_n)}{|VK_n|}-\frac{Tr(P_n\log(I-f(u))P_n)}{|VK_n|}\right|\to 0,\ n\to\infty$$

and, by using Lemma 6.4,

$$\begin{aligned}\lim_{n\to\infty}\frac{\log Z_{K_n}(u)}{|VK_n|}&=-\frac{1}{2}\lim_{n\to\infty}\frac{Tr(P_n(Q-I)P_n)}{|VK_n|}\log(1-u^2)\\ &\qquad-\lim_{n\to\infty}\frac{Tr(P_n\log(I-f(u))P_n)}{|VK_n|}\\ &=-\frac{1}{|\mathcal{F}_0|}\left(\frac{1}{2}Tr_\Gamma(Q-I)\log(1-u^2)+Tr_\Gamma(\log(I-f(u)))\right)\\ &=\frac{1}{|\mathcal{F}_0|}\ \log Z_{X,\Gamma}(u),\end{aligned}$$

from which the claim follows. □

Remark 6.7. Observe that $\frac{1}{2\alpha}<\frac{1}{d+\sqrt{d^2+2(d-1)}}<\frac{1}{\alpha}$.

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